J. Fluid Mech. (1975), *vol.* 69, *part* 1, *pp.* 97–108
Printed in Great Britain

On the flow near a weak shock wave downstream of a nozzle throat

By A. F. MESSITER AND T. C. ADAMSON
Department of Aerospace Engineering, University of Michigan, Ann Arbor

(Received 8 July 1974)

In a transonic nozzle flow in which the velocity is slightly supersonic in some neighbourhood of the nozzle throat, a shock wave may be present either very close to the throat or else somewhat further downstream. In the latter case, relatively simple series solutions in general provide an asymptotic description of the fluid motion except very close to the shock wave. These outer solutions are reviewed for symmetric two-dimensional flow, and it is shown that the shock-wave jump conditions are not satisfied. A correction is then derived in the form of an inner solution for a small region immediately behind the shock. The resulting solution exhibits the singularities in the pressure gradient, streamline curvature and shock-wave curvature which are expected to occur at the intersection of a normal shock wave and a curved wall. An extension to axisymmetric flow is also given.

1. Introduction

For steady two-dimensional transonic flow through a nozzle having a plane of symmetry, one method of solution for the velocity potential uses a series expansion postulated by Szaniawski (1965) and derived in a more systematic way by Adamson, Messiter & Richey (1974). The series proceeds in powers of a small parameter, say E, with coefficients containing powers of the transverse coordinate y and functions of the distance x along the centre-line which are determined by the wall shape. If $E \ll 1$ measures the typical variation in local Mach number within a distance from the throat of the same order as the nozzle width, the corresponding relative changes in mass flow and cross-sectional area are of order E^2, because the Mach number is near one. Thus E^2 measures the ratio of the nozzle width to the radius of curvature of the wall at the throat.

It was pointed out by Adamson *et al.* (1974) that in some cases this form of series solution is not uniformly valid as the nozzle throat is approached. As an example it was shown that the outer Szaniawski solution can be matched asymptotically with an inner similarity solution which satisfies the nonlinear transonic small disturbance equation in a small region at the throat. Adamson & Richey (1973) have shown that shock discontinuities can be incorporated in these similarity solutions, and so the example mentioned can be generalized to permit shock waves close to the throat. However, the shocks thus obtained are not exactly normal to the streamlines, and so the results are not directly applicable to inviscid nozzle flows because the replacing of a streamline by a smooth solid wall would imply a violation of the tangency condition at the wall.

The application of these ideas when a shock wave is present further downstream proceeds in a somewhat different fashion. In § 2 the series solutions are reviewed for this case. The possibility of singular behaviour at the nozzle throat is considered in a way different from that in the previous paper, and it is also shown that the shock-wave relations cannot be satisfied by these solutions. It is argued that these solutions must therefore be regarded as outer asymptotic expansions which are to be matched with an inner expansion valid very close to the shock wave. In § 3 it is shown that the non-uniform behaviour occurs on the downstream side of the shock, and an inner expansion is constructed which satisfies all the necessary conditions. The largest correction term is obtained by solution of Laplace's equation, in suitable variables, in a semi-infinite strip. A logarithmic singularity in the pressure gradient appears at the foot of the shock, as was also surmised in several previous studies. The location of the shock wave is discussed, and the extension to axisymmetric flow is outlined briefly.

2. Outer solutions and their singular behaviour

Let x and y be non-dimensional co-ordinates measured along and normal to the centre-line of a symmetric two-dimensional nozzle, with $x = 0$ at the point of minimum width and with the reference length equal to half the minimum width. We shall consider inviscid flow of a perfect gas with constant specific heats; the flow will be assumed steady, with uniform total enthalpy. If the gas velocity and the local sound speed made non-dimensional with the critical sound speed are denoted by $\mathbf{q}$ and a respectively, then

$$a^2\nabla . \mathbf{q} = \mathbf{q} . \nabla(\tfrac{1}{2}q^2), \tag{1}$$

$$\frac{a^2}{\gamma-1}+\frac{q^2}{2} = \frac{\gamma+1}{2(\gamma-1)}, \tag{2}$$

where γ is the ratio of specific heats. For a convergent–divergent nozzle, the wall shape can be expressed as

$$y = \pm[1+E^2f(x)], \tag{3}$$

where $f(0) = f'(0) = 0$. If $f''(x)$ is continuous and non-zero at the throat, we can take $f''(0) = 1$, so that E^2 is the ratio of the minimum half-width h to the radius of curvature R of the wall at $x = 0$. Upstream and downstream conditions are taken to be such that for $E^2 = h/R \ll 1$ and $x = O(1)$ the magnitude of the non-dimensional velocity is close to one. The shape of a shock wave located at $x = O(1)$ can be expressed in the form

$$x = x_s(y; E), \tag{4}$$

where $x_s = O(1)$ and $x_s' \ll 1$ for $E \ll 1$. The flow is assumed to be irrotational upstream of the shock wave, and behind the shock the vorticity remains negligible to the order of approximation considered here. A perturbation potential ϕ

will be defined by $u = 1+\phi_x$ and $v = \phi_y$, where u and v are the x and y components of $\mathbf{q}$ respectively and $\phi_x^2+\phi_y^2 \ll 1$. Equations (1) and (2) then give

$$\begin{aligned}\phi_{yy} = {} & (\gamma+1)\,\phi_x\phi_{xx}+2\phi_y\phi_{xy}+(\gamma-1)\,\phi_x\phi_{yy}+2\phi_x\phi_y\phi_{xy} \\ & +\tfrac{1}{2}(\gamma+1)\,\phi_x^2\phi_{xx}+\tfrac{1}{2}(\gamma-1)\,\phi_x^2\phi_{yy}+\tfrac{1}{2}(\gamma-1)\,\phi_y^2\phi_{xx}+\tfrac{1}{2}(\gamma+1)\,\phi_y^2\phi_{yy}. \qquad (5)\end{aligned}$$

The tangency conditions at the nozzle walls are

$$\frac{\phi_y(x,\,\pm 1 \pm E^2 f(x))}{1+\phi_x(x,\,\pm 1 \pm E^2 f(x))} = \pm E^2 f'(x). \qquad (6)$$

Following Adamson *et al.* (1974), the solutions given by Szaniawski (1965) can be derived by assuming that ϕ possesses an asymptotic expansion

$$\phi(x,y;E) \sim E\phi_1(x,y)+E^2\phi_2(x,y)+E^3\phi_3(x,y)+\dots \qquad (7)$$

valid as $E \to 0$ with x and y held fixed. Here E appears as a measure of the typical difference between the gas velocity and the sound speed at a distance $x = O(1)$ from the nozzle throat. Substitution of the assumed form (7) into the differential equation (5) and the boundary condition (6) leads to a sequence of problems which can be solved successively for $\phi_1, \phi_2, \dots$, and indicates that the interpretation of E in (7) is consistent with the definition (3). The leading terms of (5) and (6) give

$$\phi_{1yy} = 0, \quad \phi_{1y}(x,\,\pm 1) = 0 \qquad (8)$$

and so

$$\phi_1(x,y) = h_1(x). \qquad (9)$$

If higher-order terms are retained in (5) and (6), it is found that ϕ_2 and ϕ_3 satisfy the differential equations

$$\phi_{2yy} = (\gamma+1)\,h_1' h_1'', \qquad (10)$$

$$\phi_{3yy} = (\gamma+1)\,(h_1'\phi_{2xx}+h_1''\phi_{2x})+\tfrac{1}{2}(\gamma+1)\,h_1'^2 h_1''+(\gamma+1)\,h_1'\phi_{2yy} \qquad (11)$$

and the boundary conditions

$$\phi_{2y}(x,\,\pm 1) = \pm f'(x), \qquad (12)$$

$$\phi_{3y}(x,\,\pm 1) = \pm h_1'(x) f'(x). \qquad (13)$$

The solutions to (10)–(13) are

$$\phi_2(x,y) = \tfrac{1}{2} f'(x)\,y^2+h_2(x), \qquad (14)$$

$$\begin{aligned}\phi_3(x,y) = {} & \tfrac{1}{24}(\gamma+1)\,[h_1'(x) f''(x)]'\,y^4+\{(\gamma+1)\,[h_1'(x)\,h_2'(x)]' \\ & +\tfrac{1}{2}(\gamma-\tfrac{1}{2})\,h_1'(x) f'(x)\}\,y^2+h_3(x), \qquad (15)\end{aligned}$$

where

$$h_1'(x) = \pm\,[2/(\gamma+1)]^{\frac{1}{2}}\{f(x)+c_1\}^{\frac{1}{2}}, \qquad (16)$$

$$h_2'(x) = \tfrac{1}{6}(3-2\gamma)\,h_1'^2(x)-\tfrac{1}{6}f''(x)+c_2/h_1'(x). \qquad (17)$$

For $c_1 > 0$, the upper and lower signs in (16) correspond, respectively, to supersonic flows having minimum velocity near the nozzle throat and subsonic flows having maximum velocity near the throat.

If $c_1 = 0$, the terms $h_2', h_3', \ldots$, occurring in $\phi_{2x}, \phi_{3x}, \ldots$, may be singular at $x = 0$. The full solutions for h_1' and h_2' are given above, and we can easily find the singular part of h_3'. The function h_3' is determined by the boundary condition for ϕ_{4y}, which is obtained from (6) when terms of order E^4 are retained. It is seen that $\phi_{4y}(x, \pm 1)$ is bounded as $x \to 0$, and the form of h_3' as $x \to 0$ is found by satisfying this requirement. The singular part of the differential equation for ϕ_4 arises from the term $(\gamma+1)\,\phi_{2x}\phi_{2xx}$; $h_3'(x)$ appears in the term $(\gamma+1)\,(\phi_{1x}\phi_{3x})_x$. For example, if $f(x) \equiv \frac{1}{2}x^2$, then $h_1' = \pm(\gamma+1)^{-\frac{1}{2}}|x|$ and as $x \to 0$

$$\phi_{4yy} \sim \pm(\gamma+1)^{\frac{1}{2}}\,(|x|h_3')' - (\gamma+1)^2\frac{c_2^2}{x^3} \pm \tfrac{1}{2}(\gamma+1)^{\frac{3}{2}}\frac{c_2}{x^2}(\tfrac{1}{3}-y^2)\,\mathrm{sgn}\,x + \ldots. \tag{18}$$

With $h_3' \sim \mp\frac{1}{2}(\gamma+1)^{\frac{3}{2}}\,c_2^2/|x|^3$, the solutions for the velocity components u and v as $x \to 0$ are

$$u \sim 1 \pm E(\gamma+1)^{-\frac{1}{2}}|x| + E^2\left\{\pm(\gamma+1)^{\frac{1}{2}}\frac{c_2}{|x|} + \tfrac{1}{2}y^2 - \tfrac{1}{6} + \ldots\right\}$$
$$+E^3\left\{\mp\frac{(\gamma+1)^{\frac{3}{2}}c_2^2}{2|x|^3} \pm \frac{(\gamma+1)^{\frac{1}{2}}c_3}{|x|} + \ldots\right\} + \ldots, \tag{19}$$

$$v \sim E^2xy + E^3\{\pm\tfrac{1}{6}(\gamma+1)^{\frac{1}{2}}\,(y^3-y)\,\mathrm{sgn}\,x + \ldots\} + \ldots. \tag{20}$$

For $c_1 = c_2 = c_3 = 0$, two flows having continuous velocity and acceleration are possible. A strictly accelerating flow is obtained by taking the lower signs for $x < 0$ and the upper signs for $x > 0$; a decelerating flow is obtained by reversing these signs. In these cases it is seen that the second terms in $u-1$ and v are no longer small compared with the first terms if $x = O(E)$, and one might therefore question the validity of the solution for small x. The possibility of a non-uniformity at $x = 0$ is also easily inferred from the full equations (5) and (6). By introducing a stretched co-ordinate x/E^β and taking the limit as $E \to 0$ with x/E^β and y held fixed, one concludes from the boundary condition (6) that

$$\phi_y = O(E^2x) = O(E^{2+\beta}) \quad \text{and so also} \quad \phi = O(E^{2+\beta}).$$

It is then seen from the differential equation (5) that ϕ_{yy} and $\phi_x\phi_{xx}$ are of the same order, namely $O(E^3)$, if $\beta = 1$. Thus for $x = O(E)$, equation (5) gives

$$\phi_{yy} \sim (\gamma+1)\,\phi_x\phi_{xx} + o(E^3) \tag{21}$$

and so the leading term in ϕ satisfies the transonic small disturbance equation. Two solutions of (21) which satisfy the boundary condition $\phi_y(x, \pm 1) \sim \pm E^2f'(x)$ for $f(x) = \frac{1}{2}x^2$ are given by

$$u \sim 1 + E^2\{\pm(\gamma+1)^{-\frac{1}{2}}|x|/E + \tfrac{1}{2}y^2 - \tfrac{1}{6}\}, \tag{22}$$

$$v \sim E^3\{(x/E)\,y \pm \tfrac{1}{6}(\gamma+1)^{\frac{1}{2}}\,(y^3-y)\,\mathrm{sgn}\,x\}, \tag{23}$$

where the meaning of the signs is the same as before. Equations (22) and (23) describe two special cases of the similarity solutions given by Tomotika & Tamada (1950). Since the terms in (22) and (23) are exactly the largest terms in (19) and (20) as $x \to 0$, the solution given by (19) and (20) when $c_2 = c_3 = 0$ can in fact provide a correct description of the flow near $x = 0$.

If, however, $c_1 = 0$ and the same sign is used in $h_1' = \pm(\gamma+1)^{-\frac{1}{2}}|x|$ for both $x > 0$ and $x < 0$, the change from an accelerating to a decelerating flow, or vice versa, occurs rather abruptly at a speed very close to the sound speed, and it is expected that (21) may be required for small x. In these cases it may be more convenient to modify the assumed form of solution (7) by permitting c_1 to depend on E in the result for h_1' given by (16). If c_1 were replaced by $(\gamma+1)\,(Ec_2+E^2c_3+\ldots)$, equation (16) would become

$$h_1'(x) = \pm\{x^2/(\gamma+1)+2Ec_2+2E^2c_3+\ldots\}^{\frac{1}{2}}. \tag{24}$$

The terms in ϕ_2 and ϕ_3 which contain c_2 and c_3 would now be omitted, since they are the terms which would arise if h_1' as given by (24) were expanded for $E \to 0$. If $c_2 > 0$ in (24), then $E^2\phi_2$ remains much smaller than $E\phi_1$ as $x \to 0$, and so (19) and (20), with the modified form for $h_1'(x)$, remain correct as $x \to 0$. But if $c_2 = 0$ and $c_3 > 0$, this is no longer true, and the nonlinear equation (21) is needed for $x = O(E)$. The same conclusion follows for any wall shape $f(x)$ such that

$$f(x) \sim \tfrac{1}{2}x^2 \quad \text{as} \quad x \to 0.$$

At a shock wave described by $x = x_s\,(y;E)$ as in (4), another non-uniformity can be shown to occur. It is assumed tentatively that

$$x_s(y;E) \sim x_0 + Ex_1 + E^\alpha x_2(y) + \ldots, \tag{25}$$

where x_0 and x_1 are independent of y and $\alpha > 1$. If the shock wave were exactly normal to the streamlines at each point, it would follow that $\alpha = 2$; it is shown in §3 that the correct value is $\alpha = \frac{3}{2}$. The shock-polar equation relating velocity components upstream and downstream of the shock is

$$(v_d u_u - v_u u_d)^2 = \frac{(u_u^2+v_u^2-u_d u_u - v_d v_u)^2\,(u_d u_u + v_d v_u - 1)}{[2/(\gamma+1)]\,(u_u^2+v_u^2) - (u_d u_u + v_d v_u - 1)}, \tag{26}$$

where the subscripts u and d denote values immediately upstream and downstream of the shock wave respectively. This equation is derived from the more familiar form, for which $v_u = 0$, by a rotation of co-ordinates. Since $u_u \sim 1$, $v_u \ll 1$ and $v_d \ll 1$, the largest terms of (26) give

$$(v_d - v_u)^2 \sim \tfrac{1}{2}(\gamma+1)\,(u_u-u_d)^2\,(u_u u_d - 1). \tag{27}$$

At a shock wave the jump in the velocity vector is in the direction normal to the shock, and so has components in the ratio

$$(v_d - v_u)/(u_d - u_u) = -x_s'. \tag{28}$$

If we now take $u_d - u_u = O(E)$ and $x_s' = O(E^\alpha)$, with $\alpha > 1$, it follows that

$$u_u u_d = 1 + O(E^{2\alpha}). \tag{29}$$

That is, to the order required, the Prandtl relation for a normal shock wave is sufficient. Solutions given by the first two terms of (7) are assumed to be valid both upstream and downstream, with upper and lower signs chosen in (16) for $x < x_s$ and $x > x_s$ respectively. Substituting these solutions in (29), we find

$$E(u_{1u}+u_{1d}) + E^2(u_{2u}+u_{2d}+u_{1u}u_{1d}) + \ldots = 0, \tag{30}$$

where $u_1 = \phi_{1x}$ and $u_2 = \phi_{2x}$. If the terms in (30) are expanded about the approximate shock location $x = x_0$, it follows for $E \to 0$ that

$$h'_{1d}(x_0) = -h'_{1u}(x_0), \tag{31}$$

$$h'_{2d}(x_0) = -h'_{2u}(x_0) - x_1 h''_{1d}(x_0) - x_1 h''_{1u}(x_0) - h'_{1d}(x_0)\, h'_{1u}(x_0) - f''(x_0)\, y^2. \tag{32}$$

The condition (31) is satisfied for the choice of signs proposed above, and the terms proportional to x_1 in (32) sum to zero. However, the term proportional to y^2 can not be cancelled, and so (32) can not be satisfied. We conclude that the error arises because asymptotic solutions of the form (7) should be regarded as outer solutions which are not necessarily correct at points close to the shock wave. An inner solution which does satisfy all the required conditions is obtained in the next section.

3. Inner solution for the flow near the shock wave

Since the jump conditions at the shock wave are satisfied by the term in u which is of order E but not by the term of order E^2, we anticipate that an inner solution valid close to the shock wave such that the former term is constant but the latter changes rapidly with x is required. Here ϕ_{xx} is larger than elsewhere, and $\phi_x \phi_{xx}$ is no longer small in comparison with ϕ_{yy}. If we define an inner variable $x^* = (x - x_0)/E^\sigma$, the rapidly varying term in ϕ is $O(E^{2+\sigma})$, and the corresponding terms in ϕ_{xx} and ϕ_{yy} are $O(E^{2-\sigma})$ and $O(E^{2+\sigma})$ respectively. Since $\phi_x = O(E)$, the suggested balance of terms gives $\sigma = \frac{1}{2}$. The velocity components may be written in the form

$$u \sim 1 \pm E h'_{10} \pm E^{\frac{3}{2}}(\gamma+1)^{\frac{1}{2}}\, h'^{\frac{1}{2}}_{10} h''_{10} x^* + E^2\{\pm \tfrac{1}{2}(\gamma+1)\, h'_{10} h''_{10} x^{*2}$$
$$\pm x_1 h''_{10} + \tfrac{1}{2} f''_0 (y^2 - \tfrac{1}{3}) + \tfrac{1}{6}(3 - 2\gamma)\, h'^2_{10} \pm c_2/h'_{10} + u^*(x^*, y)\} + \ldots, \tag{33}$$

$$v \sim E^2 f'_0 y + E^{\frac{5}{2}}(\gamma+1)^{\frac{1}{2}}\, h'^{\frac{1}{2}}_{10}\{f''_0 x^* y + v^*(x^*, y)\} + \ldots, \tag{34}$$

where we have defined

$$x^* = (\gamma+1)^{-\frac{1}{2}}\, h'^{-\frac{1}{2}}_{10} E^{-\frac{1}{2}}(x - x_0), \tag{35}$$

$$h'_{10} = h'_{1u}(x_0) = 2^{\frac{1}{2}}\, (\gamma+1)^{-\frac{1}{2}}(f_0 + c_1)^{\frac{1}{2}} \tag{36}$$

and

$$f_0 = f(x_0), \quad f'_0 = f'(x_0), \quad f''_0 = f''(x_0), \quad h''_{10} = h''_{1u}(x_0) \quad \text{and} \quad h'''_{10} = h'''_{1u}(x_0).$$

At the shock wave $x^* = O(E^{\frac{1}{2}})$, and so the functions appearing in the shock jump conditions will be expanded about their values at $x^* = 0$. The upper and lower signs in (33) are to be taken for $x^* < 0$ and $x^* > 0$ respectively. The constant c_2 may have different values c_{2u} and c_{2d} for $x < x_0$ and $x > x_0$; an equation relating c_{2u}, c_{2d} and x_0 will be obtained below. The functions shown explicitly in (33) and (34) are just the largest terms in the expansion of the outer solution (7) as $|x - x_0| \to 0$; u^* and v^* will give the deviations from the outer solution.

If we define a term $\phi^*(x^*, y)$ in the perturbation potential by

$$u^* = \phi^*_{x^*}, \quad v^* = \phi^*_y \tag{37}$$

the differential equation (5) gives, for $E \to 0$,

$$\phi^*_{yy} = \pm \phi^*_{x^*x^*}, \tag{38}$$

where again the upper and lower signs are to be taken for $x^* < 0$ and $x^* > 0$ respectively. Substituting (33), (34) and (37) in the boundary condition (6), we find

$$\phi^*_y(x^*, \pm 1) = 0. \tag{39}$$

We obtain conditions for $|x^*| \to \infty$ by suitable asymptotic matching of the inner and outer solutions. It is assumed that there exist values of E and x such that $E \ll 1$, $|x-x_0| \ll 1$ and $|x^*| \gg 1$ for which the potential ϕ given by (7) differs from that obtained using (33) and (34) by terms much smaller than $E^{\frac{3}{2}}$. Since (38) is a wave equation for $x^* < 0$ and is Laplace's equation for $x^* > 0$, the matching conditions are satisfied to the appropriate order if we impose initial conditions for $x^* \to -\infty$ and a boundary condition for $x^* \to +\infty$, as follows:

$$\phi^*_{x^*}(x^*, y), \quad \phi^*_y(x^*, y) \to 0 \quad \text{as} \quad x^* \to -\infty, \tag{40}$$

$$\phi^*_{x^*}(x^*, y) \to 0 \quad \text{as} \quad x^* \to +\infty. \tag{41}$$

Equation (29) again is a correct approximation to the shock-polar equation (26), at least to order E^2, with u obtained by evaluating (33) at $x^* = 0$. Terms of order E^2 give

$$f''_0(y^2 - \tfrac{1}{3}) + \tfrac{1}{3}(3-2\gamma)\, h'^2_{10} + (c_{2u} - c_{2d})/h'_{10} + u^*_u + u^*_d - h'^2_{10} = 0, \tag{42}$$

where u^*_u and u^*_d are values immediately upstream and downstream of $x^* = 0$ respectively.

For $x^* < 0$, the only solution which satisfies the boundary condition (39) and the matching conditions (40) as $x^* \to -\infty$ is $u^* = v^* = 0$. Thus $u^*_u = 0$, and the Prandtl relation (42) gives

$$u^*_d = f''_0(\tfrac{1}{3} - y^2) + \tfrac{2}{3}\gamma h'^2_{10} - (c_{2u} - c_{2d})/h'_{10}, \tag{43}$$

where x_0 and c_{2d} are still unspecified.

Equations (39), (41) and (43), with u^*_d set equal to $\phi^*_{x^*}(0, y)$, give the boundary conditions needed to determine ϕ^* from (38) for $x^* > 0$ and $|y| < 1$. These boundary values must be such that the integral of the normal derivative of ϕ^*, which gives a net volume outflow, is equal to zero. Thus the integral of u^* from $y = -1$ to $y = 1$ must be the same as $x^* \to \infty$ as at $x^* = 0$. It follows from (41) and (43) that

$$c_{2d} = -\tfrac{2}{3}\gamma h'^3_{10} + c_{2u}, \tag{44}$$

$$\phi^*_{x^*}(0, y) = f''_0(\tfrac{1}{3} - y^2). \tag{45}$$

Equation (44) can be regarded as completing the determination of the downstream outer solution for ϕ_2 in terms of the corresponding upstream solution and the shock location as approximated by $x = x_0$. The downstream solution for ϕ^* which satisfies all the boundary conditions is now found to be

$$\phi^*(x^*, y) = f''_0 \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} e^{-n\pi x^*} \cos n\pi y. \tag{46}$$

The approximate location of the shock wave can be determined if the upstream flow is known and one quantity downstream is also given. For example, if u_2 is known at some value of x, then c_{2d} is also known; h'_{10} is given by equation (44), and x_0 therefore can be found. Since, however, the inner solution provides only a local correction, it is not really needed for the derivation of (44). An alternative derivation can be given which permits a more direct interpretation and which also suggests a simplified procedure for the calculation of x_1. The entropy jump at a normal shock wave is

$$\frac{s_d - s_u}{R} = \frac{1}{\gamma-1}\log\frac{p_d}{\rho_d^\gamma} = \frac{1}{\gamma-1}\log\left\{\left(\frac{2\gamma M_u^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\right)\left(\frac{(\gamma-1)M_u^2+2}{(\gamma+1)M_u^2}\right)^\gamma\right\}, \quad (47)$$

where s is the specific entropy, R is the gas constant, p and ρ are the pressure and density made non-dimensional with sonic values upstream of the shock wave, M is the Mach number and the subscripts u and d again refer to conditions immediately upstream and downstream of the shock wave respectively. Equation (2) provides an expression for M_u in terms of u_u, and the first terms of the outer solution for u_u are found from (9), (14), (16) and (17). After these substitutions have been made, expansion of (47) gives

$$(s_d - s_u)/R \sim \tfrac{2}{3}E^3\gamma(\gamma+1)h_{10}'^3 + E^4\gamma(\gamma+1)h_{10}'^2\{f_0''(y^2-\tfrac{1}{3}) - \tfrac{2}{3}\gamma h_{10}'^2 + 2x_1 h_{10}'' + 2c_{2u}/h'_{10}\} + \ldots, \quad (48)$$

where the expansion $u_{1u} = h'_{10} + Ex_1 h''_{10} + \ldots$ has also been used. To these orders the entropy is constant along lines y = constant, which are approximately coincident with the streamlines. Hence (47) and (48) may be regarded as expressing s and p/ρ^γ approximately as functions of y at any location downstream of the shock wave. If the term of order E^3 in the entropy or total pressure were specified, then x_0 would be determined. If instead (47) is used to eliminate p in (2), with $a^2 = p/\rho$, an expansion for ρu can be obtained:

$$\rho u \sim 1 - E^2\frac{\gamma+1}{2}u_1^2 + E^3(\gamma+1)\left(\frac{3-2\gamma}{6}u_1^3 - u_1u_2\right) + E^4(\gamma+1)\left\{-u_1u_3 + \frac{u_2^2}{2} + (\tfrac{3}{2}-\gamma)u_1^2u_2 + \tfrac{1}{2}\left(1-\frac{\gamma}{2}\right)(\gamma-\tfrac{1}{2})u_1^4\right\} + \ldots - \frac{s_d - s_u}{R}. \quad (49)$$

The mass flow is then found to be, for $x > x_0$,

$$\int_0^{1+E^2f(x)} \rho u\, dy \sim 1 + E^2\left(f - \frac{\gamma+1}{2}h_1'^2\right) - E^3(\gamma+1)(c_{2d} + \tfrac{2}{3}\gamma h_{10}'^3) + \ldots. \quad (50)$$

At the throat $x = 0$, since $\tfrac{1}{2}(\gamma+1)u_1^2 = c_1$ and the term $(s_d - s_u)/R$ is absent, the corresponding result is

$$\int_0^1 \rho u\, dy = 1 - E^2c_1 - E^3(\gamma+1)c_{2u} + \ldots. \quad (51)$$

These two expressions must of course be identical. The terms of order E^2 agree, according to (16), and if we equate the terms of order E^3, we recover (44). By retaining terms of order E^4, we would obtain an additional equation relating x_1 and u_3. The pressure is found by combining (47), (48) and (2):

$$p \sim 1 - E\gamma u_1 - E^2\gamma u_2 + E^3\{-\gamma u_3 - \tfrac{1}{2}\gamma(\gamma+1)u_1^3\} + \ldots. \quad (52)$$

At some downstream location $x = x_r$ where the walls are parallel and

$$f' = f'' = \ldots = 0,$$

we might suppose that the pressure $p = 1+\epsilon$ is given where $\epsilon = E\gamma h_1'(x_r)+\ldots$ is a small positive number consistent with (16), so that $c_1 \geqslant 0$. Terms of order $E^2, E^3, \ldots,$ in (52) may be specified arbitrarily; for example, we may set these terms equal to zero. Then x_0 and x_1 are found by requiring that the mass flow be the same at $x = 0$ and at $x = x_r$. From these considerations it is seen, as anticipated, that no term of order $E^{\frac{3}{2}}$ is required in the expression for the shock-wave shape given by (25), whereas in general a constant term of order E does appear.

The shock-wave slope $x_s'(y)$ is found from (28). After substituting (33), (34) and (46), one finds $\alpha = \frac{3}{2}$, consistent with the original assumption $\alpha > 1$, and

$$x'(y;E) \sim -E^{\frac{3}{2}}\frac{2}{\pi^2}(\gamma+1)^{\frac{1}{2}}h_0'^{-\frac{1}{2}}f_0'' \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin n\pi y. \tag{53}$$

The term ϕ^* in the perturbation potential and the shock-wave contour given by $x = x_s(y)$ have second derivatives which are singular at the intersections of the shock wave and the nozzle walls. Differentiating (46) twice and rewriting in complex form, one finds

$$\phi^*_{x^*x^*} - i\phi^*_{x^*y} = \frac{4}{\pi}f_0'' \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\{\exp[-\pi(x^*+iy)]\}^n. \tag{54}$$

Replacing $(-1)^n$ by $e^{\pm in\pi}$ and summing the series, one next obtains

$$\phi^*_{x^*x^*} - i\phi^*_{x^*y} = (-4/\pi)f_0''\log\{1-\exp[-\pi(x^*+i(y\mp 1))]\}. \tag{55}$$

For $|x^*+i(y\mp 1)| \to 0$,

$$\phi^*_{x^*x^*} - i\phi^*_{x^*y} \sim (-4/\pi)f_0''\log\{\pi[x^*+i(y\mp 1)]\}. \tag{56}$$

The real and imaginary parts contribute respectively to the pressure gradient and to the streamline curvature. As $x^* \to 0$ and $y \to \pm 1$,

$$p_x \sim E\gamma h_0'' + E^{\frac{3}{2}}\gamma(\gamma+1)^{-\frac{1}{2}}h_0'^{-\frac{1}{2}}(2/\pi)f_0''\log\{\pi^2[x^{*2}+(y\mp 1)^2]\} \tag{57}$$

and so p_x has a logarithmic singularity such that a weak but rapid expansion occurs along the surface immediately downstream of the shock wave; just upstream of the shock $p_x \sim -E\gamma h_0'' + O(E^2)$. The streamline curvature downstream of the shock as $x^* \to 0$ and $y \to \pm 1$ is

$$v_x \sim E^2\left\{\pm f_0'' + \frac{4}{\pi}f_0''\tan^{-1}\frac{y\mp 1}{x^*}\right\}, \tag{58}$$

where the upper and lower signs refer to the upper and lower nozzle walls respectively. It is seen that the limiting value of v_x at the intersection of the shock wave and the nozzle wall depends on the path of approach to the point. For example, the value is consistent with the body curvature if first $y \to \pm 1$ and then $x^* \to 0$, but differs if first $x^* \to 0$ and then $y \to \pm 1$. Since $p_y \sim -v_x$, along the shock one finds $p_y \sim -yf_0''$ for $-1 < y < 1$ just upstream and $p_y \sim \pm f_0''$ as $y \to \pm 1$ just downstream. Thus near the walls the shock-wave relations considered alone would lead to a change of sign in p_y and therefore in v_x, and the wall boundary condition would be violated. The solution to the differential equations shows that the boundary condition still can be satisfied, but that the solution is singular. Finally, the shock-wave curvature as $y \to \pm 1$ is found to be

$$x_s'' \sim \frac{f_0''}{\pi}\left(\frac{\gamma+1}{h_0'}\right)^{\frac{1}{2}} 2E^{\frac{1}{2}} \log\left[\pi(1 \mp y)\right] \tag{59}$$

and has a logarithmic singularity. Since $x_s'' < 0$ near $y = \pm 1$, the shock wave curves slightly upstream from the normal to each wall, if $f_0'' > 0$.

The form of singularity shown in (57)–(59) has also been proposed by Gadd (1960), Oswatitsch & Zierep (1960) and Ferrari & Tricomi (1968, p. 358). Noting the contradiction arising at a curved wall because of the sign change in p_y predicted by the normal-shock relations, they postulated a local solution which does permit both the shock-wave jump conditions and the wall boundary condition to be satisfied. For an airfoil in steady motion at a slightly subsonic speed, with negligible boundary-layer effects, it seems reasonable to expect that these solutions correctly describe the flow at the foot of a shock wave which terminates an embedded supersonic region. The present derivation provides an example in which their assumed local solutions can be obtained as part of the solution to a boundary-value problem for the flow in a larger region. These singularities in the derivatives can be removed by an analysis which takes into account the interaction of the inviscid flow and the boundary layer along the nozzle walls. For an unseparated boundary layer, a solution describing small perturbations on an undisturbed velocity profile would be coupled with a solution to the nonlinear transonic small disturbance equation in a region having dimensions small in comparison with $E^{\frac{1}{2}}$ for sufficiently large Reynolds number.

The outer and inner representations in the form given by (7), (33) and (34) can be combined to give a composite expansion valid throughout the region $x = O(1)$ as follows:

$$\phi(x,y;E) \sim E\phi_1(x,y) + E^2\phi_2(x,y) + E^{\frac{3}{2}}(\gamma+1)^{\frac{1}{2}} h_{10}'^{\frac{1}{2}}\phi^*(x^*,y) + \dots, \tag{60}$$

where ϕ^* is now defined to be zero for $x^* < 0$. The factor $\{E(\gamma+1)h_{10}'\}^{\frac{1}{2}}$ is just the first approximation to $(M_u^2 - 1)^{\frac{1}{2}}$. Equation (60), together with the solutions for ϕ_1, ϕ_2 and ϕ^* given previously, permits calculation of terms $O(E^2)$ and $O(E^{\frac{3}{2}})$ in u and v respectively, with c_{2d} expressed in terms of some flow quantity evaluated at a reference location downstream of the shock wave. For small $|x - x_0|$, expansion of the first two terms in (60) gives the functions shown explicitly in (33) and (34), and so (60) reduces to the inner solution. For $x^* \to \infty$, ϕ^* is exponentially small, and for $x^* \to -\infty$, ϕ^* is zero, so that the outer solution is recovered if $|x^*| \gg 1$. If $c_1 > 0$ or if $c_1 = c_{2u} = c_{3u} = \dots = 0$, the composite solution also remains correct at $x = 0$. If $c_1 = 0$ and $c_{2u} > 0$, the result is again valid near $x = 0$ if c_1 is replaced by $(\gamma+1)Ec_{2u}$ as proposed in §2. If, however, $c_1 = c_{2u} = 0$ and $c_{3u} > 0$, a modification is required since the solution is not correct in detail near the throat.

A similar description can be given for an axisymmetric nozzle flow with a shock wave located at a distance $x = O(1)$ downstream of the throat. The nozzle wall is at $r = 1+E^2f(x)$, where r is the radial co-ordinate. The velocity components now are ϕ_x and ϕ_r, and ϕ_{yy} is replaced by $\phi_{rr}+r^{-1}\phi_r$ when $\nabla.\mathbf{q}$ is evaluated in (1). An outer solution is found as before, in terms of a series proceeding in powers of E as in (7). Again it is found that the shock-wave jump conditions can not be satisfied and so an inner solution is introduced, in the manner of (33)–(36). The results can be shown in a more compact form if a composite solution is written down directly. One finds

$$\phi(x,r;E) \sim Eh_1(x)+E^2\{\tfrac{1}{2}f'(x)\,r^2+h_2(x)\}+E^{\frac{5}{2}}(\gamma+1)^{\frac{1}{2}}h_{10}'^{\frac{1}{2}}\phi^*(x^*,r)+\ldots, \tag{61}$$

where

$$h_1'(x) = \pm 2(\gamma+1)^{-\frac{1}{2}}\{f(x)+c_1\}^{\frac{1}{2}}, \tag{62}$$

$$h_2'(x) = \tfrac{1}{6}(3-2\gamma)\,h_1'^2(x)-\tfrac{1}{4}f''(x)+c_2/h_1'(x), \tag{63}$$

$$x^* = E^{-\frac{1}{2}}(\gamma+1)^{-\frac{1}{4}}h_{10}'^{-\frac{1}{4}}(x-x_0). \tag{64}$$

Again $h_{10}' = h_{1u}'(x_0)$, $f_0 = f(x_0)$, etc. and $h_2'(x)$ has been found using the boundary condition for ϕ_3. The term ϕ^* is understood to be zero for $x^* < 0$, and satisfies Laplace's equation with boundary condition $\phi_r^*(x^*,1) = 0$ for $x^* > 0$. The solution which is bounded at $r = 0$ is given in terms of Bessel functions by the series

$$\phi^*(x^*,r) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x^*}J_0(\lambda_n r), \tag{65}$$

where the λ_n are the zeros of J_0':

$$J_0'(\lambda_n) = 0. \tag{66}$$

Following the same steps as in the two-dimensional case, one finds

$$c_{2d} = -\tfrac{2}{3}\gamma h_{10}'^3+c_{2u} \quad \text{and} \quad \phi_{x^*}^*(0,r) = (\tfrac{1}{2}-r^2)f_0''.$$

The boundary condition at the shock wave is then satisfied if

$$-\lambda_n a_n\int_0^1 rJ_0^2(\lambda_n r)\,dr = f_0''\int_0^1 r(\tfrac{1}{2}-r^2)\,J_0(\lambda_n r)\,dr. \tag{67}$$

Using various identities and recursion formulae, one finds

$$a_n = 4f_0''/\lambda_n^3 J_0(\lambda_n). \tag{68}$$

The approximate location of the shock wave can be found as in the plane case from the requirement that $\phi_{x^*}^*$ should contribute nothing to the mass flow. At the intersection of the shock wave with the nozzle wall, the second derivatives of (65) are expected to show the same singularities as in the two-dimensional case. For $\lambda_n \gg 1$, the terms in the series (65) as $x^* \to 0$ and $r \to 1$ are approximately

$$\frac{1}{\lambda_n^3}\frac{J_0(\lambda_n r)}{J_0(\lambda_n)}e^{-\lambda_n x^*} \sim \frac{(-1)^n}{n^3}e^{-n\pi x^*}\cos n\pi r. \tag{69}$$

Since the singular behaviour of the series depends on the form of the terms for large n, it follows that the second derivatives of (65) and (46) do, in fact, have the same singularities. Thus, all the general features of the flow are the same as in the plane case.

This work was supported by the Office of Naval Research, Project SQUID, under contract N00014-67-A-0226-0005. The authors gratefully acknowledge this support.

REFERENCES

Adamson, T. C., Messiter, A. F. & Richey, G. K. 1974 *Arch. Mech. Stos.* **6**, 617–628.
Adamson, T. C. & Richey, G. K. 1973 *J. Fluid Mech.* **60**, 363–382.
Ferrari, C. & Tricomi, F. G. 1968 *Transonic Aerodynamics.* Academic.
Gadd, G. E. 1960 *Z. angew. Math. Phys.* **11**, 51–58.
Oswatitsch, K. & Zierep, J. 1960 *Z. angew Math. Mech.* **40**, 143–144.
Szaniawski, A. 1965 *Arch. Mech. Stos.* **17**, 79–85.
Tomotika, J. & Tamada, K. 1950 *Quart. Appl. Math.* **7**, 381–397.